SYDNEY GRAMMAR SCHOOL

NAME

MATHS MASTER

CANDIDATE NUMBER

2023 **Trial HSC Examination**

Form VI Mathematics Extension 2

Tuesday 8th August 2023
8:40am

General Instructions

- Reading time — 10 minutes
- Working time — 3 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 100

Section I (10 marks) Questions 1 – 10
- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (90 marks) Questions 11 – 16
- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- Your name and master should only be written on this page.
- Write your candidate number on this page, on each booklet and on the multiple choice sheet.
- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.
- Place everything inside this question booklet.

Checklist
- Reference sheet
- Multiple-choice answer sheet
- 6 booklets per boy
- Candidature: 81 pupils

Writer: RR

Section I

Questions in this section are multiple-choice.
Record the single best answer for each question on the provided answer sheet.

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1. Which of the following is the correct expression for $\displaystyle\int \frac{1}{\sqrt{1-9x^2}}\,dx$?

   (A) $\dfrac{1}{3}\sin^{-1}\dfrac{x}{3}+C$

   (B) $3\sin^{-1}\dfrac{x}{3}+C$

   (C) $\dfrac{1}{3}\sin^{-1}3x+C$

   (D) $3\sin^{-1}3x+C$

2. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ satisfy $(\underset{\sim}{a}-\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b})=0$, which of the following must be true?

   (A) $\underset{\sim}{a}=\pm\underset{\sim}{b}$

   (B) $|\underset{\sim}{a}|=|\underset{\sim}{b}|$

   (C) $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$

   (D) $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$

3. Consider the statement:

   '$\forall\, a\in\mathbf{Z},\ \exists\, b\in\mathbf{Z}$ such that $\frac{a}{b}\in\mathbf{Z}$'

   What does this statement mean?

   (A) Every integer has a factor.

   (B) There exists an integer which divides every integer.

   (C) Every integer is also a rational number.

   (D) There exists a rational number corresponding to each pair of integers.

4. Which of the following is the correct expression for $\displaystyle\int x\cos x\,dx$?

   (A) $x\sin x-\sin x+C$

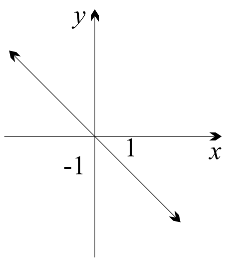
   (B) $x\sin x+\cos x+C$

   (C) $x\cos x-\cos x+C$

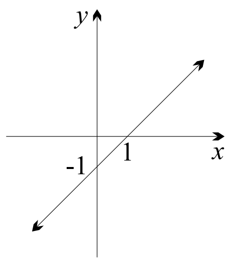
   (D) $x\cos x+\sin x+C$

5. Which of the following curves satisfies the equation $\text{Arg}(z-1) = \text{Arg}(z+i)$?

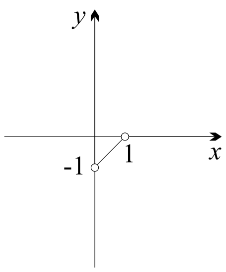
(A)



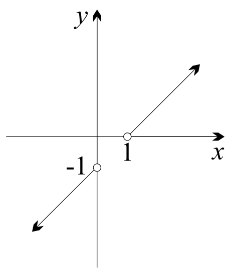
(B)



(C)



(D)



6. Consider the statement:

‘If n is even, then $n^3 + n$ is even.’

Which of the following are true?

(A) The contrapositive of the original statement but NOT the converse

(B) The converse of the original statement but NOT the contrapositive

(C) Both the contrapositive and the converse of the original statement

(D) Neither the contrapositive nor the converse of the original statement

7. What is the angle made between the line $\underset{\sim}{r} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where $\lambda \in \mathbf{R}$, and the yz-plane?

You may assume $a, b, c > 0$.

(A) $\sin^{-1}\left(\dfrac{a}{\sqrt{a^2+b^2+c^2}}\right)$

(B) $\sin^{-1}\left(\dfrac{b}{\sqrt{a^2+b^2+c^2}}\right)$

(C) $\tan^{-1}\left(\dfrac{b}{\sqrt{a^2+c^2}}\right)$

(D) $\tan^{-1}\left(\dfrac{c}{\sqrt{a^2+b^2}}\right)$

8. Which of the following gives the correct expression for $\displaystyle\int \frac{\sin x}{\sin x + \cos x}\,dx$?

   (A) $\frac{1}{2}(x + \ln|\sin x + \cos x|) + C$

   (B) $\frac{1}{2}(x - \ln|\sin x + \cos x|) + C$

   (C) $-\frac{1}{2}(x + \ln|\sin x + \cos x|) + C$

   (D) $-\frac{1}{2}(x - \ln|\sin x + \cos x|) + C$

9. If z is a complex number such that $|z+i| = 5$, what is the maximum value of $|z^2 - 1|$?

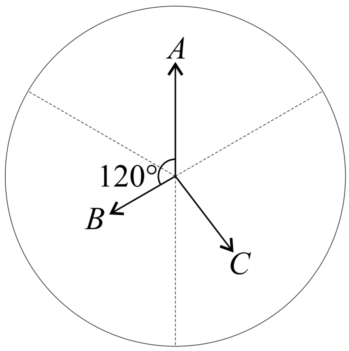
   (A) 17

   (B) 25

   (C) 35

   (D) 37

10. Alpha, Bravo, and Charlie are competing in a three-way tug-of-war. They each hold a rope, and the three ropes are joined together at the centre of a circle. A competitor wins if they can pull the intersection point into their sector (see below):



    Alpha and Bravo pull with a force of a and b newtons respectively at an angle of $120°$ to one another. Charlie pulls with a force of c newtons and pulls at an angle directly opposing the sum of the two other forces.

    Given this scenario, which of the following statements is true?

    (A) If $c > \frac{a+b}{2}$, then Charlie wins.

    (B) If $c > ab$, then Charlie wins.

    (C) If Charlie wins, then $c^2 > ab$.

    (D) If Charlie wins, then $c > a + b$.

End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (15 marks) Start a new answer booklet. **Marks**

(a) Let $z = \sqrt{3} + i$ and $w = 1 - i$.

(i) Express $\dfrac{z}{w}$ in the form $a + bi$. 1

(ii) Express z and w in modulus-argument form. 1

(iii) Hence determine the exact value of $\tan \frac{5\pi}{12}$. 2

(b) Evaluate $\displaystyle\int_0^1 \frac{4x+1}{x+1}\, dx$. 2

(c) Consider the statement: 2

'If $n^2 - 3n + 2$ is odd, then n is odd.'

Prove that the statement is true using contraposition.

(d) (i) Use the result $e^{i\theta} = \cos\theta + i\sin\theta$ to show $e^{ni\theta} + e^{-ni\theta} = 2\cos n\theta$. 1

(ii) By expanding $\left(e^{i\theta} + e^{-i\theta}\right)^4$, prove that $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$. 2

(iii) Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$. 1

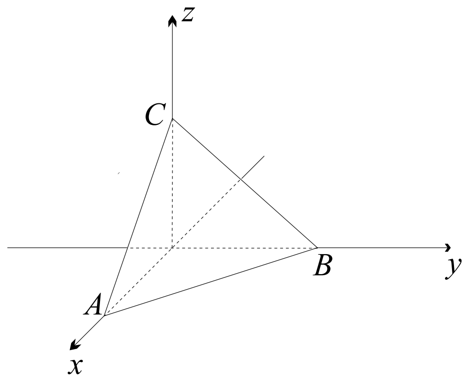
(e) A particle experiences acceleration according to the equation $\ddot{x} = 2v$, where v is the particle's velocity in metres per second. It begins at the origin with a velocity of 1 m/s. 3

How far does it travel in the first two seconds? Write your answer correct to the nearest metre.

The paper continues on the next page

QUESTION TWELVE (15 marks) Start a new answer booklet. **Marks**

(a) Let ABC be a triangle with vertices $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,k)$ for some constant $k>0$. This is shown below:



(i) Express $\overrightarrow{AC}$ and $\overrightarrow{BC}$ as column vectors. 1

(ii) Find the value of k that will make $\angle ACB = 45^\circ$. Write your answer correct to 2 decimal places. 2

(b) (i) Find the constants A, B and C such that $\dfrac{x^2+8x+16}{x^3+4x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$. 2

(ii) Hence find $\displaystyle\int \frac{x^2+8x+16}{x^3+4x}\,dx$. 2

(c) The acceleration of an object moving in a straight line is given by $\ddot{x} = -\dfrac{1}{1+x}$, where x metres is the displacement of the object from the origin. The object is initially at the origin with a velocity of $2\,\text{m/s}$.

(i) Find v^2 as a function of x. 2

(ii) Hence find where the object will first come to a rest, and justify whether it must turn back around or not. 2

(d) (i) Prove using contradiction that $\sqrt{6}$ is irrational. 2

(ii) Hence prove that it is not possible for both $\sqrt{2n}$ and $\sqrt{3n}$ to be rational for any positive integer n. 2

The paper continues on the next page

QUESTION THIRTEEN (15 marks) Start a new answer booklet. **Marks**

(a) Consider the points $A(5,0,2)$, $B(2,3,-4)$ and $C(8,-3,5)$.

(i) Find an equation for the line AB. $\boxed{1}$

(ii) Show that the point C is not on the line AB. $\boxed{1}$

(iii) Find the point on the line AB which is closest to point C. $\boxed{2}$

(b) A particle moves in a straight line and its motion satisfies the equation

$$v^2 = 32 + 8x - 4x^2,$$

where x is the particle's displacement from the origin in metres, and v is its velocity in metres per second.

(i) Show that the motion is simple harmonic. $\boxed{1}$

(ii) Find the maximum acceleration $|\ddot{x}|$ achieved by the particle. $\boxed{2}$

(c) Use the substitution $u = \sqrt{e^x - 1}$ to evaluate $\displaystyle\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$. $\boxed{3}$

(d) (i) Use deMoivre's Theorem to show that $\boxed{2}$

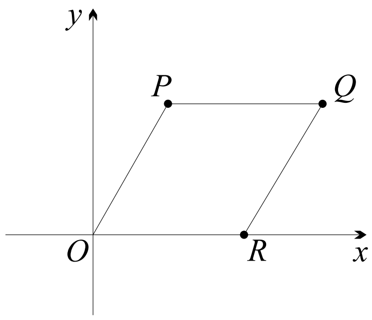
$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta\,.$$

(ii) By considering the solutions of $\sin 5\theta = 0$, find the exact value of $\sin^2 \frac{\pi}{5}$. $\boxed{3}$

The paper continues on the next page

QUESTION FOURTEEN (15 marks) Start a new answer booklet. **Marks**

(a) Let $z = \cos\theta + i\sin\theta$ be some complex number with $0 < \theta < \frac{\pi}{2}$ and let the points P, Q, and R represent the complex numbers z, $z+1$, and 1 respectively, as shown below: 4



By considering the quadrilateral $OPQR$, or otherwise, show that

$$\frac{2z}{z+1} = 1 + i\tan\frac{\theta}{2}\,.$$

(b) Let x and y be positive real numbers.

(i) Prove that $x + \dfrac{1}{x} \geq 2$, for all $x > 0$. 1

(ii) Hence show that 2

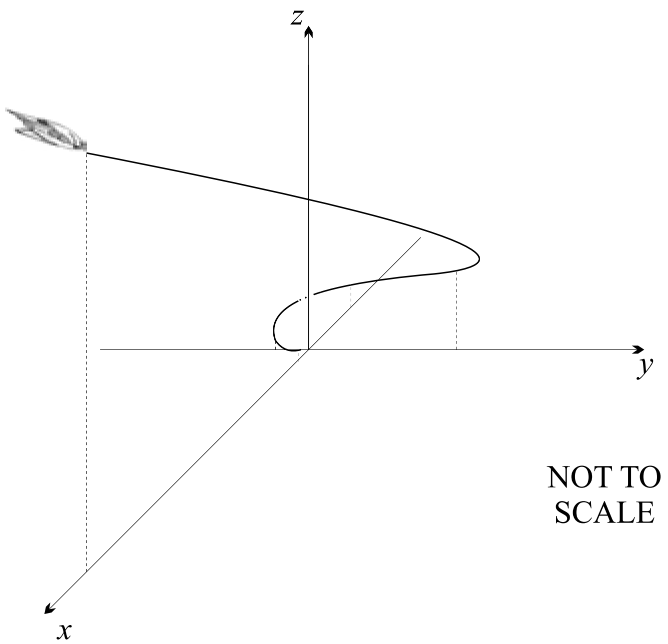
$$\frac{1+x^2}{y} + \frac{1+y^2}{x} \geq 4\,.$$

(c) By using the substitution $t = \tan\frac{x}{2}$, find $\displaystyle\int \frac{1}{1+\sin x - \cos x}\,dx$. 3

The paper continues on the next page

QUESTION FOURTEEN (Continued)

(d) A trained falcon is soaring above a crowd before gliding down to land on its trainer. In order to maintain constant eye contact without having to move its eyes or head, it glides along a path known as a logarithmic spiral, as shown below:



The position of the falcon, t seconds after beginning its descent, is given by

$$\underset{\sim}{r} = \begin{pmatrix} 20e^{-t}\cos t \\ 20e^{-t}\sin t \\ 10e^{-t} \end{pmatrix},$$

where the units of the components are in metres.

(i) Find an expression for the velocity $\underset{\sim}{v}$ of the falcon at any time t, and hence calculate its speed as it begins its descent. $\boxed{2}$

(ii) Show that the angle between $\underset{\sim}{r}$ and $\underset{\sim}{v}$ is constant. $\boxed{3}$

The paper continues on the next page

QUESTION FIFTEEN (16 marks) Start a new answer booklet. **Marks**

(a) (i) Find the three cube roots of $2+2i$ and plot these roots on the Argand diagram. 3

(ii) Given that the three roots sum to zero, show that 2

$$\cos\left(\frac{\pi}{12}\right) - \sin\left(\frac{\pi}{12}\right) = \frac{1}{\sqrt{2}}.$$

(b) Consider the polynomial $P(z) = z^4 + kz^2 + 1$, for some $k \in \mathbf{R}$.

(i) Show that if $z = \omega$ is a zero of $P(z)$, then so are $-\omega$, $\overline{\omega}$, and ω^{-1}. 2

(ii) If $P(\omega) = 0$ and $|\omega| \neq 1$, justify that either $\mathrm{Re}(\omega) = 0$ or $\mathrm{Im}(\omega) = 0$. 3

(c) The HMS Dreadnought was a British battleship in the early 20th century and is the only battleship to have ever sunk a submarine. As part of its artillery, it could launch torpedoes of mass 500 kg at an initial speed of 10 m/s into the water. While in the water, the torpedo would self-propel with a constant driving force of 2000 N and approach a terminal velocity of 20 m/s.

For this question, you can assume that the torpedo travels horizontally through the water at a constant depth and that the torpedo experiences resistance from the water proportional to the square of its velocity.

(i) Show that the displacement of the torpedo from the ship satisfies the equation 2

$$\ddot{x} = 4 - \frac{v^2}{100}.$$

(ii) How long would it take the torpedo to hit a target 1000 m away? Write your answer correct to the nearest second. 4

The paper continues on the next page

QUESTION SIXTEEN (14 marks) Start a new answer booklet. **Marks**

(a) Let n be an positive odd integer and define recursively the double factorial: $\boxed{3}$

$$n!! = \begin{cases} n \times (n-2)!!, & \text{for } n \geq 3 \\ 1, & \text{for } n = 1 \end{cases}$$

Note that $n!!$ is not the same as $(n!)!$.

Prove using induction that

$$n!! = \frac{n!}{2^{\frac{n-1}{2}} \times \left(\frac{n-1}{2}\right)!}$$

for all positive odd integers.

(b) Consider the definite integral

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx \,,$$

where m and n are non-negative integers.

(i) Using the substitution $x = \frac{\pi}{2} - u$, show that $I_{m,n} = I_{n,m}$. $\boxed{1}$

(ii) Prove that $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$, for $n \geq 2$. $\boxed{3}$

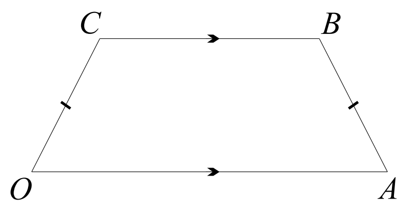
(iii) Using the previous results from both part (a) and part (b), show that $\boxed{4}$

$$\int_0^{\frac{\pi}{2}} \sin^{2k} x \cos^{2l} x \, dx = \frac{(2k)!(2l)!}{k!\, l!\, (k+l)!} \times \frac{\pi}{2^{2k+2l+1}}$$

for all non-negative integers k and l.

(c) An isosceles trapezium $OABC$ is a quadrilateral with sides OA and BC parallel but NOT equal in length, and sides OC and AB equal in length but not parallel. $\boxed{3}$

This is shown below:



Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{c} = \overrightarrow{OC}$.

Show, using vector methods, that the diagonals OB and AC are equal in length.

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- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

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Extension 2 2023 Trial - Solutions

Q1. C   Note $\frac{d}{dx}\sin^{-1}(3x) = \frac{3}{\sqrt{1-(3x)^2}}$

Q2. B $(a-b)\cdot(a+b) = |a|^2 - |b|^2 = 0$

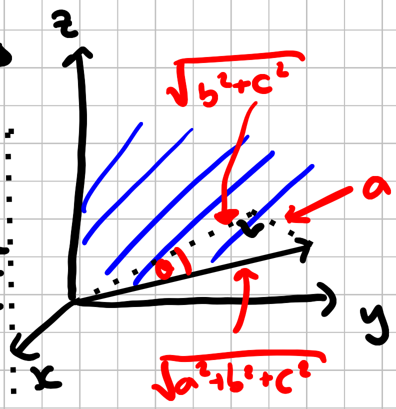
Q3. A   "For all a, there is a b which divides it."

Q4. B $\frac{d}{dx}(x\sin x + \cos x) = x\cos x - \sin x + \sin x$

Q5. D   Same direction from 1 and $-i$.

Q6. A   The original statement is true but not the converse

Q7. A   See diagram to right

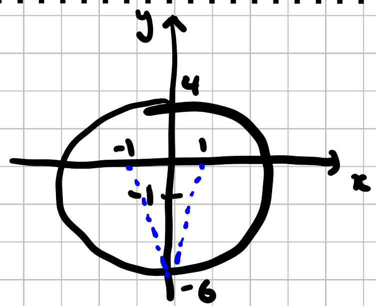


Q8. B   Best done by testing...

$$\frac{d}{dx}\left(\ln|\sin x + \cos x|\right) = \frac{\cos x - \sin x}{\sin x + \cos x}$$

$$= 1 - 2\frac{\sin x}{\cos x}$$

Q9. D $|z+i| = 5$ so z sits on circle:



$|z^2-1| = |z+1||z-1|$

which is maximised by $z = 6i$

$= \sqrt{37} \times \sqrt{37}$

Q10. C   The others are easily seen to be false by example:

(A) If $a=100$, $b=0$, $c=51$: Charlie should lose

(B) As above

(D) If $a=100$, $b=100$, $c=101$: Charlie should win

To see why C is true: Charlie wins precisely when $c^2 > a^2+b^2-2ab\cos 60°$

$$= a^2+b^2-ab$$
$$= (a-b)^2 + ab$$

Hence if Charlie wins, $c^2 > ab$

Q11 a) i) $z = \sqrt{3} + i$, $w = 1 - i$

$$\frac{z}{w} = \frac{\sqrt{3}+i}{1-i} \times \frac{1+i}{1+i}$$

$$= \frac{(\sqrt{3}-1) + (\sqrt{3}+1)i}{2}$$

$$= \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i \checkmark$$

ii) $z = 2\operatorname{cis}\frac{\pi}{6}$, $w = \sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right) \checkmark$

iii) $$\frac{z}{w} = \frac{2}{\sqrt{2}}\operatorname{cis}\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$$

$$= \frac{2}{\sqrt{2}}\operatorname{cis}\left(\frac{5\pi}{12}\right) \checkmark$$

Equating real parts: $\frac{2}{\sqrt{2}}\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2}$ (2)

Equating imaginary parts: $\frac{2}{\sqrt{2}}\sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2}$ (1)

(1) $\div$ (2) : $\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \checkmark$

$$= 2+\sqrt{3}$$

11 b) $$\int_0^1 \frac{4x+1}{x+1}\,dx = \int_0^1 \frac{4x+4}{x+1} - \frac{3}{x+1}\,dx$$

$$= \Big[4x - 3\ln|x+1|\Big]_0^1 \checkmark$$

$$= 4 - 3\ln 2 \checkmark$$

11c) The contrapositive statement is
'If n is even, then n^2-3n+2 is even'
(✓ For clear reference/understanding of contrapositive)

So suppose $n=2k$ for some $k \in \mathbb{Z}$.

Then $n^2-3n+2 = (2k)^2 - 3(2k)+2$
$= 4k^2-6k+2$
$= 2(2k^2-3k+1)$ ✓

which is even.

Hence, by contraposition, the original statement is true.

11d) i) $e^{in\theta} + e^{-in\theta} = [\cos(n\theta) + i\sin(n\theta)] + [\cos(-n\theta) + i\sin(-n\theta)]$ ✓

$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$ {as $\cos\theta$ even & $\sin\theta$ odd}

$= 2\cos n\theta$

ii) $(e^{i\theta} + e^{-i\theta})^4 = e^{i4\theta} + 4e^{i2\theta} + 6 + 4e^{-i2\theta} + e^{-i4\theta}$ ✓

$= (e^{i4\theta} + e^{-i4\theta}) + 4(e^{i2\theta} + e^{-i2\theta}) + 6$

so $(2\cos\theta)^4 = 2\cos 4\theta + 4(2\cos 2\theta) + 6$ ✓

$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\Rightarrow \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

11 d) iii) $\int_0^{\pi/2} \cos^4\theta = \int_0^{\frac{\pi}{2}} \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$= \left[\frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}x\right]_0^{\frac{\pi}{2}}$

$= \frac{3\pi}{16}$ ✓

11e)

$\ddot{x} = 2v$

$\frac{dv}{dt} = 2v$

$\frac{dt}{dv} = \frac{1}{2v}$

integrating, $t = \frac{1}{2}\ln|v| + C$

when $t=0$, $v=1$ so $C=0$

$t = \frac{1}{2}\ln(v)$ ✓

$v = e^{2t}$

$\frac{dx}{dt} = e^{2t}$

integrating, $x = \frac{1}{2}e^{2t} + C$ ✓

when $t=0$, $x=0$ so $C = -\frac{1}{2}$

$x = \frac{1}{2}(e^{2t} - 1)$

alt:

$v\frac{dv}{dx} = 2v$

$\frac{dv}{dx} = 2$

$v = 2x + C$

when $x=0$, $v=1$

so $C=1$

$v = 2x+1$

$\frac{dx}{dt} = 2x+1$

$t = \frac{1}{2}\ln(2x+1) + C$

when $x=0$, $t=0$

so $C = 0$

$\Rightarrow x = \frac{1}{2}(e^{2t}-1)$.

Letting $t=2$,

$x \doteqdot 27$m ✓ (nearest metre)

12a) i) $\vec{AC} = \begin{pmatrix} -1 \\ 0 \\ k \end{pmatrix}$ and $\vec{BC} = \begin{pmatrix} 0 \\ -1 \\ k \end{pmatrix}$ ✓

ii) If $\angle ACB = 45°$, then

$$\frac{\vec{AC} \cdot \vec{BC}}{|\vec{AC}|\,|\vec{BC}|} = \cos 45°$$

so $$\frac{k^2}{\sqrt{k^2+1} \cdot \sqrt{k^2+1}} = \frac{1}{\sqrt{2}}$$ ✓

$$\Rightarrow \quad \sqrt{2}k^2 = k^2+1$$

$$(\sqrt{2}-1)k^2 = 1$$

$$k^2 = \frac{1}{\sqrt{2}-1}$$

$$k \doteqdot 1.55$$ ✓

b) i) let $$\frac{x^2+8x+16}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

then $$x^2+8x+16 = A(x^2+4) + (Bx+c)x$$

Equating constants: $4A = 16 \Rightarrow A = 4$ ✓

Equating coefficients of x: $C = 8$

Equating coefficients of x^2: $A + B = 1 \Rightarrow B = -3$ ✓

ii) $$\int \frac{x^2+8x+16}{x^3+4x}\,dx = \int \frac{4}{x} - \frac{3x}{x^2+4} + \frac{8}{x^2+4}\,dx$$

$$= 4\ln|x| - \frac{3}{2}\ln|x^2+4|$$ ✓

$$+ 4\tan^{-1}\frac{x}{2} + C$$ ✓

12c) i) $$\ddot{x} = -\frac{1}{1+x}$$

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -\frac{1}{1+x}$$

$$\tfrac{1}{2}v^2 = -\ln|1+x| + \frac{c}{2} \checkmark$$

$$v^2 = C - 2\ln|1+x|$$

when $x=0$, $v=2$ so $C=4$

$$v^2 = 4 - 2\ln|1+x| \checkmark$$

ii) Letting $v^2=0$ gives

$$2\ln|1+x| = 4$$

$$1+x = e^2 \quad \text{(note } v>0 \text{ initially so } x>0\text{)}$$

$$x = e^2-1 \checkmark$$

when $x = e^2-1$

$$\ddot{x} = -\frac{1}{e^2}$$

$$<0$$

so the object will turn around as v becomes negative. ✓

12d) i) Suppose, by way of contradiction, that $\sqrt{6}$ is rational.

I.e. $\sqrt{6} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with no common factors.

Then $6 = \frac{a^2}{b^2}$

so $6b^2 = a^2$

Since LHS is even, RHS is even and hence a is even. ✓

Let $a = 2k$

so $6b^2 = (2k)^2$

$= 4k^2$

$\Rightarrow \quad 3b^2 = 2k^2$

12 d i) (cont.) $3b^2 = 2k^2$

Now RHS is even, so LHS is even $\Rightarrow$ b is even.

But then a & b are both even, so they share a common factor, which is a contradiction. ✓

Thus, by contradiction, $\sqrt{6}$ is irrational.

ii) Now suppose, by way of contradiction, that both $\sqrt{2n}$ and $\sqrt{3n}$ are rational.

I.e. $\sqrt{2n} = \frac{p}{q}$ and $\sqrt{3n} = \frac{r}{s}$ ✓ for $p, q, r, s \in \mathbb{Z}$

but then $\sqrt{2n} \times \sqrt{3n} = \frac{pr}{qs}$

$$\sqrt{6}\, n^2 = \frac{pr}{qs}$$

$$\sqrt{6} = \frac{pr}{n^2 qs}$$

which implies $\sqrt{6}$ is rational. ✓

This is a contradiction of part i), and hence $\sqrt{2n}$ and $\sqrt{3n}$ cannot both be rational.

13a) $A(5,0,2)$, $B(2,3,-4)$, $C(8,-3,5)$

i) $\underset{\sim}{r} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$

$= \begin{pmatrix}5\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}-3\\3\\-6\end{pmatrix}$ ✓

ii) Suppose $\begin{pmatrix}8\\-3\\5\end{pmatrix} = \begin{pmatrix}5\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}-3\\3\\-6\end{pmatrix}$

then $8 = 5 - 3\lambda \Rightarrow \lambda = -1$

but equating z coefficients gives

$5 = 2 - 6\lambda$

$\Rightarrow 6\lambda = -3$

$\Rightarrow \lambda = -\frac{1}{2}$ ✓

since these do not agree, C is not on AB.

iii) $\overrightarrow{AC} = \begin{pmatrix}3\\-3\\3\end{pmatrix}$

$\text{proj}_{\overrightarrow{AB}}\overrightarrow{AC} = \dfrac{\begin{pmatrix}3\\-3\\3\end{pmatrix}\cdot\begin{pmatrix}-3\\3\\-6\end{pmatrix}}{\left\|\begin{pmatrix}-3\\3\\-6\end{pmatrix}\right\|^2} \cdot \begin{pmatrix}-3\\3\\-6\end{pmatrix}$

$= \dfrac{-36}{54}\begin{pmatrix}-3\\3\\-6\end{pmatrix}$

$= \begin{pmatrix}2\\-2\\4\end{pmatrix}$ ✓

so closest point is $\overrightarrow{OA} + \begin{pmatrix}2\\-2\\4\end{pmatrix} = \begin{pmatrix}7\\-2\\6\end{pmatrix}$ ✓

13b) i)

$$v^2 = 32 + 8x - 4x^2$$
$$\tfrac{1}{2}v^2 = 16 + 4x - 2x^2$$
$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 4 - 4x$$
$$\ddot{x} = -4(x-1) \checkmark$$ (must be factorised)

Hence the motion is simple harmonic (with centre of motion $x=1$)

ii) Let $v=0$, then

$$32 + 8x - 4x^2 = 0$$
$$\Rightarrow \quad x^2 - 2x - 8 = 0$$
$$\Rightarrow \quad (x-4)(x+2) = 0$$

so the particle is at rest when $x=-2$ or $x=4$ ✓

At these points $|\ddot{x}| = 4|x-1|$
$$= 12 \checkmark$$

so max acceleration is $12\ m/s^2$.

13 c).

$$\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$$
$$= \int_0^1 u \cdot \frac{2u}{u^2+1}\, du$$
$$= \int_0^1 2 - \frac{2}{u^2+1}\, du \checkmark$$
$$= \left[2u - 2\tan^{-1} u\right]_0^1$$
$$= 2 - \frac{\pi}{2} \checkmark$$

Let $u = \sqrt{e^x - 1}$

so $du = \dfrac{e^x}{2\sqrt{e^x-1}}\, dx$ ✓

$$= \frac{u^2+1}{2u}\, dx$$

so $\dfrac{2u}{u^2+1}\, du = dx$

when $x=0$, $u=0$
$x=\ln 2$, $u=1$

13d) i) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ {De Moivre's Theorem}

and $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$
$- 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta$
$+ i\sin^5\theta$ ✓

Equating imaginary parts:

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \checkmark$$
$$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$ as req^{d}.

ii) Note that $\sin\left(5 \times \frac{\pi}{5}\right) = \sin\pi$
$= 0$

so $5\sin\frac{\pi}{5} - 20\sin^3\frac{\pi}{5} + 16\sin^5\frac{\pi}{5} = 0$

clearly $\sin\frac{\pi}{5} \neq 0$, so divide through to get

$$16\sin^4\frac{\pi}{5} - 20\sin^2\frac{\pi}{5} + 5 = 0 \checkmark$$
$$16\left(\sin^2\frac{\pi}{5}\right)^2 - 20\left(\sin^2\frac{\pi}{5}\right) + 5 = 0$$

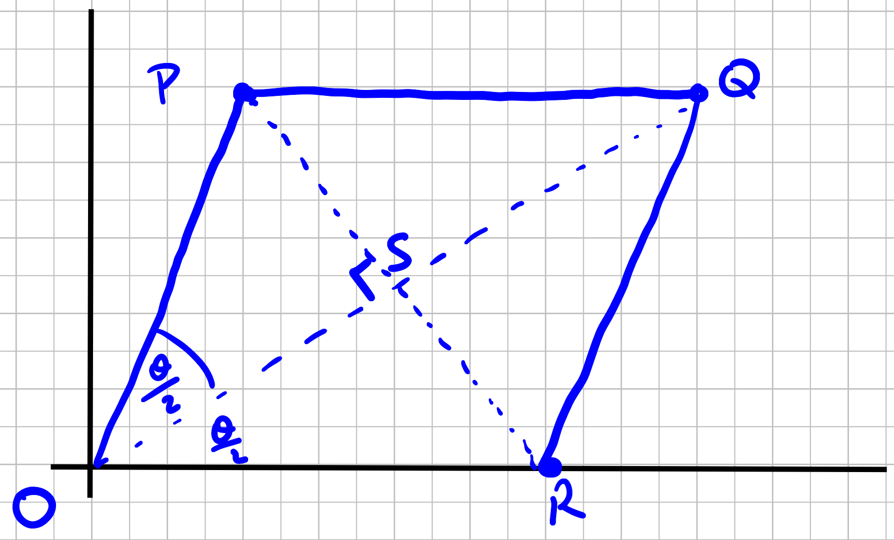
so $\sin^2\frac{\pi}{5} = \frac{20 \pm \sqrt{400-320}}{32}$ ✓

$$= \frac{5 \pm \sqrt{5}}{8}$$

and note $\sin^2\frac{\pi}{5} < \sin^2\frac{\pi}{4} = \frac{1}{2}$

so $\sin^2\frac{\pi}{5} = \frac{5-\sqrt{5}}{8}$ ✓

14a)



Note that $|z| = 1$ so OPQR is a rhombus as it is a parallelogram with adjacent sides equal.

Hence $\arg(z+1) = \frac{\theta}{2}$ (diagonal bisects vertex)

so $\arg\left(\frac{2z}{z+1}\right) = \arg(2z) - \arg(z+1)$

$= \theta - \frac{\theta}{2}$

$= \frac{\theta}{2}$.

and letting OQ & PR intersect at S

we know $OQ \perp PR$

and OQ bisects PR

$\Bigg\}$ (diagonals of a rhombus)

so $|OS| = \cos\frac{\theta}{2}$

$\Rightarrow |z+1| = 2\cos\frac{\theta}{2}$

hence $\left|\frac{2z}{z+1}\right| = \frac{|2z|}{|z+1|}$

$= \frac{1}{\cos\frac{\theta}{2}}$

so $\frac{2z}{z+1} = \frac{1}{\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

$= 1 + i\tan\frac{\theta}{2}$ as required.

14b) i) RTP: $x + \frac{1}{x} \geq 2$ for all $x > 0$

$$LHS - RHS = x - 2 + \frac{1}{x}$$
$$= \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 ✓$$
$$\geq 0$$

Hence $LHS - RHS \geq 0$

$\Rightarrow LHS \geq RHS$

ii) Since $x + \frac{1}{x} \geq 2$, it follows

$x^2 + 1 \geq 2x$ as $x > 0$.

Hence $$\frac{1+x^2}{y} + \frac{1+y^2}{x} \geq \frac{2x}{y} + \frac{2y}{x} ✓$$
$$= 2\left(\frac{x}{y} + \frac{y}{x}\right)$$
$$\geq 2 \times 2 ✓ \text{ (from part i)}$$
$$= 4.$$

Q14c). $\int \frac{1}{1+\sin x - \cos x}\, dx$ Let $t = \tan\frac{x}{2}$

$2\tan^{-1} t = x$

$\frac{2\, dt}{1+t^2} = dx$

$$= \int \frac{1}{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} ✓$$
$$= \int \frac{2\, dt}{2t + 2t^2}$$
$$= \int \frac{1}{t+t^2}\, dt$$
$$= \int \left(\frac{1}{t} - \frac{1}{1+t}\right) dt ✓$$
$$= \ln|t| - \ln|1+t| + C$$
$$= \ln\left(\tan\frac{x}{2}\right) - \ln\left(1 + \tan\frac{x}{2}\right) + C ✓$$

14 d)

$$\underset{\sim}{r} = \begin{pmatrix} 20e^{-t}\cos t \\ 20e^{-t}\sin t \\ 10e^{-t} \end{pmatrix}$$

$$\underset{\sim}{v} = \begin{pmatrix} -20e^{-t}\cos t - 20e^{-t}\sin t \\ -20e^{-t}\sin t + 20e^{-t}\cos t \\ -10e^{-t} \end{pmatrix} \checkmark$$

when $t=0$, $\underset{\sim}{v} = \begin{pmatrix} -20 \\ 20 \\ -10 \end{pmatrix}$

$$|\underset{\sim}{v}| = 30 \text{ m/s} \checkmark$$

ii) $\underset{\sim}{r} \cdot \underset{\sim}{v} = \left(-400e^{-2t}\cos^2 t - 400e^{-2t}\sin t\cos t\right)$

$+ \left(-400e^{2t}\sin^2 t + 400e^{-2t}\sin t\cos t\right)$

$+ \left(-100e^{-2t}\right)$

$= -500e^{-2t} \checkmark$

$$|\underset{\sim}{r}|^2 = 400e^{-2t}\cos^2 t + 400e^{-2t}\sin^2 t + 100e^{-2t}$$

$$= 500e^{-2t}$$

$$|\underset{\sim}{r}| = 10\sqrt{5}\, e^{-t}$$

$|\underset{\sim}{v}|^2 = \left(400e^{-2t}\cos^2 t + 800e^{-2t}\cos t\sin t + 400e^{-2t}\sin^2 t\right)$

$+ \left(400e^{-2t}\sin^2 t - 800e^{-2t}\cos t\sin t + 400e^{-2t}\cos^2 t\right)$

$+ 100e^{-2t}$

14d) ii) (cont.)

$$|\underset{\sim}{v}|^2 = 900e^{-2t}$$

$$|\underset{\sim}{v}| = 30e^{-t}$$ ✓

Let θ be the angle between $\underset{\sim}{r}$ & $\underset{\sim}{v}$

then
$$\cos\theta = \frac{\underset{\sim}{r}\cdot\underset{\sim}{v}}{|\underset{\sim}{r}|\,|\underset{\sim}{v}|}$$

$$= \frac{-500e^{-2t}}{(10\sqrt{5}e^{-t})(30e^{-t})}$$

$$\cos\theta = \frac{-\sqrt{5}}{3}$$ ✓

$$\doteqdot 138^\circ$$

(The falcon looks at a constant angle of 42°).

15a) i) Let $z = r \operatorname{cis}\theta$

and suppose $z^3 = 2 + 2i$

then $(r \operatorname{cis}\theta)^3 = 2\sqrt{2} \operatorname{cis}\frac{\pi}{4}$

$\Rightarrow \quad r^3 \operatorname{cis} 3\theta = \sqrt{2} \operatorname{cis}\frac{\pi}{4}$ {by DeMoivre's theorem} ✓

Hence $r^3 = 2\sqrt{2}$

$\Rightarrow r = \sqrt{2}$

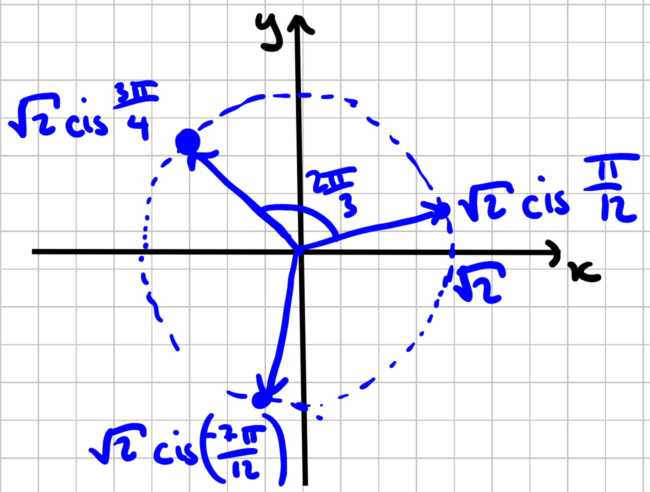
and $3\theta = \frac{\pi}{4} + 2k\pi$

$\theta = \frac{\pi}{12} + \frac{2k\pi}{3}$

Taking $k = 0, \pm 1$ gives

$z = \sqrt{2} \operatorname{cis}\left(\frac{-7\pi}{12}\right), \sqrt{2} \operatorname{cis}\left(\frac{\pi}{12}\right), \sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right)$ ✓

ii)



✓

Let $\omega = \sqrt{2} \operatorname{cis}\frac{\pi}{12}$, then the sum of roots are

$\sqrt{2} \operatorname{cis}\left(-\frac{7\pi}{12}\right) + \sqrt{2} \operatorname{cis}\left(\frac{\pi}{12}\right) + \sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right)$

$= \sqrt{2}\,\omega\left(\operatorname{cis}\left(-\frac{2\pi}{3}\right) + 1 + \operatorname{cis}\left(\frac{2\pi}{3}\right)\right)$

$= \sqrt{2}\,\omega\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i + 1 - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \sqrt{2}\,\omega \times 0$

$= 0$ ✓

15 a) iii) Adding real parts:

$$\sqrt{2}\cos\frac{\pi}{12} + \sqrt{2}\cos\frac{3\pi}{4} + \sqrt{2}\cos\frac{-7\pi}{12} = 0 \quad ✓$$

$$\cos\frac{\pi}{12} - \frac{1}{\sqrt{2}} + \cos\frac{7\pi}{12} = 0$$

$$\cos\frac{\pi}{12} - \cos\frac{5\pi}{12} = \frac{1}{\sqrt{2}}$$

$$\cos\frac{\pi}{12} - \sin\left(\frac{\pi}{2} - \frac{5\pi}{12}\right) = \frac{1}{\sqrt{2}}$$

$$\text{so } \cos\frac{\pi}{12} - \sin\frac{\pi}{12} = \frac{1}{\sqrt{2}}. \quad ✓$$

15b) i) $P(z) = z^4 + kz^2 + 1$

Suppose $P(\omega) = 0$;

then
$$P(-\omega) = (-\omega)^4 + k(-\omega)^2 + 1$$
$$= \omega^4 + k\omega^2 + 1$$
$$= 0$$

and
$$P(\bar{\omega}) = (\bar{\omega})^4 + k(\bar{\omega})^2 + 1$$
$$= \overline{\omega^4} + k\overline{\omega^2} + 1$$
$$= \overline{\omega^4 + k\omega^2 + 1}$$
$$= 0$$

(or appeal to conjugate roots theorem)

and
$$P\left(\frac{1}{\omega}\right) = \left(\frac{1}{\omega}\right)^4 + k\left(\frac{1}{\omega}\right)^2 + 1$$
$$= \frac{1 + k\omega^2 + \omega^4}{\omega^4}$$
$$= 0$$

so all are roots

✓ for two

✓✓ for three.

15b) ii) Since $P(z) = z^4 + kz^2 + 1$,

the product of zeroes is 1.

but $\omega \times (-\omega) \times \bar{\omega} \times \frac{1}{\omega} = -\omega\bar{\omega}$

$= -|\omega|^2$

≤ 0

So these cannot be the four zeroes. ✓

(In fact, you may note that

$\omega, -\omega, \bar{\omega}, \frac{1}{\bar{\omega}}, -\bar{\omega}, -\frac{1}{\omega}, \frac{1}{\bar{\omega}}, -\frac{1}{\bar{\omega}}$

are all zeroes, so some must be doubled up)

Thus at least two of $\omega, -\omega, \bar{\omega}, \frac{1}{\omega}$ are equal.

If $|\omega| \neq 1$ then

$\frac{1}{\omega}$ cannot be the same as $\omega, \bar{\omega}$, or $-\omega$. ✓

So $\omega = -\omega$ (1)

$\Rightarrow 2\omega = 0$

$\Rightarrow \omega = 0$ (not possible)

$\omega = \bar{\omega}$ (2)

$\Rightarrow \omega - \bar{\omega} = 0$

$\Rightarrow Im(\omega) = 0$

or $-\omega = \bar{\omega}$ (3)

$\Rightarrow \omega + \bar{\omega} = 0$

$\Rightarrow Re(\omega) = 0$

✓

15c). i) $F = 2000 - mkv^2$

$m\ddot{x} = 2000 - mkv^2$

$\ddot{x} = 4 - kv^2$

alt. $F = 2000 - kv^2$

$m\ddot{x} = 2000 - kv^2$

$\ddot{x} = 4 - \frac{kv^2}{500}$

✓

max speed = 20 m/s so $\ddot{x} = 0$ when $v = 20$

$\Rightarrow 0 = 4 - 400k$

$\Rightarrow k = \frac{1}{100}$

hence $\ddot{x} = 4 - \frac{v^2}{100}$ ✓

ii) ① v against x:

$$v\frac{dv}{dx} = 4 - \frac{v^2}{100}$$

$$\frac{100v}{400 - v^2}\frac{dv}{dx} = 1$$

$$\int \frac{100v}{400 - v^2}\,dv = \int dx$$

$$-50\log|400 - v^2| = x + C$$

when $x = 0$, $v = 10$

so $C = -50\log 300$

$$\Rightarrow x = 50\log\left|\frac{300}{400 - v^2}\right| \checkmark$$

when $x = 1000$

$$\frac{300}{400 - v^2} = e^{20}$$

15c) ii) (cont.)

$$400 - v^2 = 300e^{-20}$$

$$v^2 = 400 - 300e^{-20}$$

$$v = \sqrt{400 - 300e^{-20}}$$

② v against t :

$$\ddot{x} = 4 - \frac{v^2}{100}$$

$$\frac{dv}{dt} = \frac{400 - v^2}{100}$$

$$\frac{100}{400 - v^2} \frac{dv}{dt} = 1$$

$$\int \frac{100}{400 - v^2}\, dv = \int dt$$

$$\frac{100}{40} \int \frac{1}{20 + v} + \frac{1}{20 - v}\, dv = t + C$$

$$\frac{5}{2} \log \left| \frac{20 + v}{20 - v} \right| = t + C$$

when $t = 0$, $v = 10$

so $C = \frac{5}{2} \log |3|$

Hence $t = \frac{5}{2} \log \left| \frac{20 + v}{60 - 3v} \right|$

③ Combining:

Letting $v = \sqrt{400 - 300e^{-20}}$

gives $t = 51.438$ seconds

$\doteqdot 51$ seconds

16a) RTP:

$$n!! = \frac{n!}{2^{\frac{n-1}{2}} \times \left(\frac{n-1}{2}\right)!} \quad \text{for } n \text{ odd}$$

$\boxed{A}$ when $n=1$

LHS $= 1!!$
$= 1$

RHS $= \dfrac{1!}{2^0 \times 0!}$
$= 1$

so LHS=RHS

Thus the result holds for $n=1$ ✓

$\boxed{B}$ Assume the result holds for $n=k$, for k odd.

Now for $n=k+2$,

$$(k+2)!! = (k+2) \times k!!$$

$$= (k+2) \times \frac{k!}{2^{\frac{k-1}{2}} \times \left(\frac{k-1}{2}\right)!} ✓ \quad \left\{\begin{array}{l}\text{By our} \\ \text{assumption}\end{array}\right\}$$

$$= \frac{(k+2) \times (k+1) \times k!}{(k+1) \times 2^{\frac{k-1}{2}} \times \left(\frac{k-1}{2}\right)!}$$

$$= \frac{(k+2)!}{2 \times \left(\frac{k+1}{2}\right) \times 2^{\frac{k-1}{2}} \times \left(\frac{k-1}{2}\right)!}$$

$$= \frac{(k+2)!}{2^{\frac{k+1}{2}} \times \left(\frac{k+1}{2}\right)!} ✓$$

Thus the result holds for $n=k+2$ whenever it holds for $n=k$

$\boxed{C}$ So by parts A & B, the result holds for all $n \geq 1$ odd.

16b) i) Let $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$

Let $x = \frac{\pi}{2} - u$

$dx = -du$

when $x = 0$, $u = \frac{\pi}{2}$

$x = \frac{\pi}{2}$, $u = 0$

so $I_{m,n} = \int_{\frac{\pi}{2}}^{0} \sin^m\left(\frac{\pi}{2} - u\right) \cos^n\left(\frac{\pi}{2} - u\right) (-du)$

$= \int_0^{\frac{\pi}{2}} \cos^m(u) \sin^n(u) \, du$ ✓

$= \int_0^{\frac{\pi}{2}} \sin^n(u) \cos^m(u) \, du$

$= I_{n,m}$

ii) $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$

$= \int_0^{\frac{\pi}{2}} (\cos^{n-1} x)(\sin^m x \cos x \, dx)$

Let $u = \cos^{n-1} x$

$du = (n-1)\cos^{n-2} x \cdot (-\sin x) \, dx$

$dv = \sin^m x \cos x \, dx$

$v = \frac{1}{m+1} \sin^{m+1} x$ ✓

$= \left[\frac{1}{m+1} \cos^{n-1} x \sin^{m+1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{n-1}{m+1} \sin^{m+1} x \left(\cos^{n-2} x \cdot -\sin x\right) dx$

$= [0 - 0] + \frac{n-1}{m+1} \int_0^{\frac{\pi}{2}} \sin^{m+2} x \cos^{n-2} x \, dx$

$= \frac{n-1}{m+1} \int_0^{\frac{\pi}{2}} \sin^m x \left(1 - \cos^2 x\right) \cos^{n-2} x \, dx$ ✓

16b) ii) (cont.)

$$I_{m,n} = \frac{n-1}{m+1}\left[I_{m,n-2} - I_{m,n}\right]$$

$$(m+1)I_{m,n} = (n-1)I_{m,n-2} - (n-1)I_{m,n}$$

$$(m+n)I_{m,n} = (n-1)I_{m,n-2}$$

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2} \quad \text{as required.}$$

✓

iii) $$I_{2k,2\ell} = \frac{(2\ell-1)}{(2k+2\ell)} I_{2k,2\ell-2} \quad \text{(by part ii)}$$

$$= \frac{(2\ell-1)}{2(k+\ell)} \times \frac{(2\ell-3)}{2(k+\ell-1)} I_{2k,2\ell-4}$$

$$= \frac{(2\ell-1)}{2(k+\ell)} \times \frac{(2\ell-3)}{2(k+\ell-1)} \times \cdots \times \frac{1}{2(k+1)} I_{2k,0}$$ ✓

$$I_{2k,0} = I_{0,2k} \quad \text{(by part i)}$$

$$= \frac{(2k-1)}{2k} \times \frac{(2k-3)}{2(k-1)} \times \cdots \times \frac{1}{2} I_{0,0}$$

so $$I_{2k,2\ell} = \frac{(2\ell-1)!! \times (2k-1)!!}{2^{k+\ell}\,(k+\ell)!} I_{0,0}$$ ✓

$$= \frac{(2\ell-1)!}{2^{\ell-1}\times(\ell-1)!} \times \frac{(2k-1)!}{2^{k-1}\times(k-1)!} \times \frac{1}{2^{k+\ell}\times(k+\ell)!}$$ ✓

$$\times I_{0,0}$$

{using 16 a}

16b) iii) (cont.)

$$I_{2k,2\ell} = \frac{(2\ell)!}{2^{\ell} \times \ell!} \times \frac{(2k)!}{2^{k} \times k!} \times \frac{1}{2^{k+\ell} \times (k+\ell)!} \times I_{0,0}$$

$$= \frac{(2\ell)!\,(2k)!}{(\ell)!\,(k)!\,(k+\ell)!} \times \frac{I_{0,0}}{2^{2k+2\ell}}$$

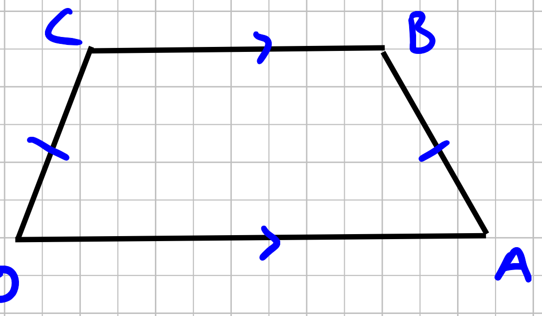
$$I_{0,0} = \int_0^{\frac{\pi}{2}} dx$$

$$= \frac{\pi}{2}$$

so $I_{2k,2\ell} = \dfrac{(2\ell)!\,(2k)!}{\ell!\,k!\,(k+\ell)!} \times \dfrac{\pi}{2^{2k+2\ell+1}}$ as req^{d}.

✓

16c) Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{c} = \overrightarrow{OC}$



$\overrightarrow{CB} \parallel \overrightarrow{OA}$ so

$\overrightarrow{CB} = k\mathbf{a}$ for some $k \neq 1$.

Then $\overrightarrow{AB} = -\mathbf{a} + \mathbf{c} + k\mathbf{a}$

$$= (k-1)\mathbf{a} + \mathbf{c}$$

$|\overrightarrow{OC}| = |\overrightarrow{AB}|$ so $|\overrightarrow{OC}|^2 = |\overrightarrow{AB}|^2$

$$|\mathbf{c}|^2 = \left((k-1)\mathbf{a} + \mathbf{c}\right) \cdot \left((k-1)\mathbf{a} + \mathbf{c}\right)$$
$$= (k-1)^2|\mathbf{a}|^2 + 2(k-1)\mathbf{a}\cdot\mathbf{c} + |\mathbf{c}|^2$$

$$\Rightarrow \quad 0 = (k-1)^2|\mathbf{a}|^2 + 2(k-1)\mathbf{a}\cdot\mathbf{c}$$

$(k-1) \neq 0$

so $\quad (k-1)|\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{c} = 0$

$$\Rightarrow \quad 2\mathbf{a}\cdot\mathbf{c} = (1-k)|\mathbf{a}|^2$$

Now $\overrightarrow{OB} = \mathbf{c} + k\mathbf{a}$

and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$$|\overrightarrow{OB}|^2 = |\mathbf{c}|^2 + 2k\,\mathbf{a}\cdot\mathbf{c} + k^2|\mathbf{a}|^2$$
$$= |\mathbf{c}|^2 + k(1-k)|\mathbf{a}|^2 + k^2|\mathbf{a}|^2$$
$$= |\mathbf{c}|^2 + k|\mathbf{a}|^2$$

$$|\overrightarrow{AC}|^2 = |\mathbf{c}|^2 - 2\mathbf{a}\cdot\mathbf{c} + |\mathbf{a}|^2$$
$$= |\mathbf{c}|^2 - (1-k)|\mathbf{a}|^2 + |\mathbf{a}|^2$$
$$= |\mathbf{c}|^2 + k|\mathbf{a}|^2$$
$$= |\overrightarrow{OB}|^2$$

so $|\overrightarrow{AC}| = |\overrightarrow{OB}|$ as required.